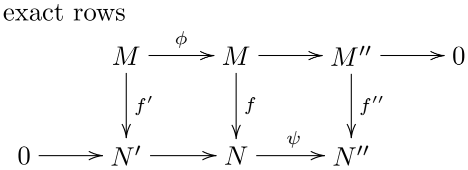
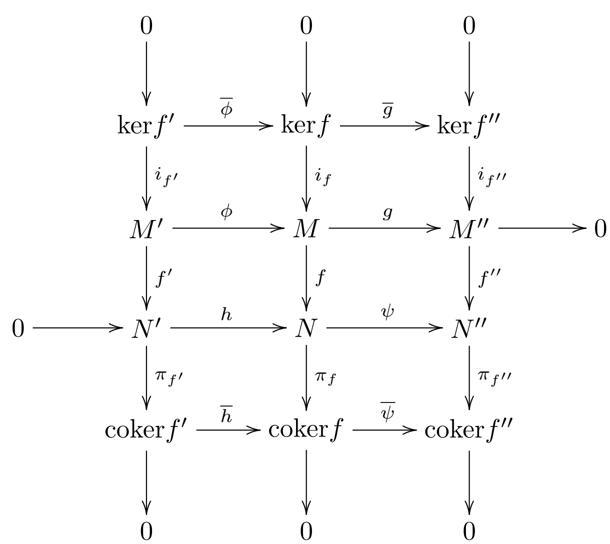
The Snake Lemma

Most graduate students in mathematics will be asked to prove the snake lemma at least once (if not multiple times) over the course of their studies. It is extremely difficult to find a complete proof that does not skip steps or simply instruct the reader to complete parts of the proof as an exercise. Here is an attempt at a thorough proof using modules. This can easily be generalized to abelian groups (as they are simply $\mathbb{Z}$-modules), or whatever the reader requires. In this lemma, an overline (or bar) over a map denotes the induced map, and over an element denotes the equivalence class of that element. (For example, $\overline{x} \in \mathrm{coker} f$ means the equivalence class $x + \mathrm{im} f$ for the element x.) The statement of the lemma is as follows:

Let R be a commutative ring with identity. Given a commutative diagram of R-modules with exact rows



there exists a map δ in the induced diagram



where i are the canonical inclusion maps, π are the canonical projection maps, and $\delta : \ker f'' \to \operatorname{coker} f'$ such that the sequence

$$\ker f' \xrightarrow{\overline{\phi}} \ker f \xrightarrow{\overline{g}} \ker f'' \xrightarrow{\delta} \operatorname{coker} f' \xrightarrow{\overline{h}} \operatorname{coker} f \xrightarrow{\overline{\psi}} \operatorname{coker} f''$$

is exact, and all sections of the diagram commute.

We can augment this statement, and we will do so, but first we prove what we have already stated.

Firstly, we show exactness of

$$0 \to \ker f \xrightarrow{i_f} M \xrightarrow{f} N \xrightarrow{\pi_f} \operatorname{coker} f \to 0$$

Since i_f is the inclusion map, it is canonically injective, and π_f is the projection map, so it is canonically surjective. Also, by definition of $\ker f$, $f \circ i_f = 0$, and by the definition of $\operatorname{coker} f$, $\pi_f \circ f = 0$. Certainly by definition, $\ker f \subset \operatorname{im} i_f$, and also by definition, $\ker \pi_f \subset \operatorname{im} f$. Thus all columns of the diagram are exact.

Next we show that $\overline{\phi}, \overline{g}, \overline{h}, \overline{\psi}$ are well defined and that the entire diagram commutes. Let $a = b$ in $\ker f'$, i.e. $a - b = 0$ in $\ker f'$. $i_{f'}$ is injective, so $i_{f'}(a-b) = 0 \in M'$. ϕ is well defined, so $\phi \circ i_{f'}(a-b) = 0 \in M$. i_f is injective, so $\phi \circ i_{f'}(a-b) = 0 = i_f(0)$. Then $\overline{\phi}$ sends $a - b = 0$ to 0 in $\ker f$, so $\overline{\phi}$ is well defined ($\overline{\phi}(a) = \overline{\phi}(b)$) and the top left section of the diagram commutes. The exact same procedure can be followed for the top right section.

Now let $\overline{a} = \overline{b}$ in $\operatorname{coker} f'$. $\pi_{f'}$ is surjective, so $\overline{a} - \overline{b} = \overline{0} = \pi_{f'}(a' - b')$ for $a', b' \in N'$. Then $h(a' - b') = 0 \in N$, so $\pi_f \circ h(a' - b') = \overline{0} \in \operatorname{coker} f$. Then $\overline{h}$ sends $\overline{a} - \overline{b} = \overline{0}$ to $\overline{0}$ in $\operatorname{coker} f$, so $\overline{h}$ is well defined, and the bottom left section of the diagram commutes. The same procedure can be followed for the bottom right section.

Thus all maps in the diagram are defined (except δ which will be shown later), and the diagram commutes. Now we will show exactness.

Exact at $\ker f$:

1. $\overline{g} \circ \overline{\phi} = 0$: Let $a \in \ker f'$. Then by commutativity, $i_f \circ \overline{\phi}(a) = \phi \circ i_{f'}(a)$. Applying g, we get 0 by exactness of the row, so $g \circ i_f \circ \overline{\phi}(a) = 0$. $i_{f''}$ is injective, so by commutativity of the diagram, $\overline{g} \circ \overline{\phi} = 0$.

2. $\ker \overline{g} \subset \overline{im}(\overline{\phi})$: Let $b \in \ker \overline{g}$. Then $\overline{g}(b) = 0 \in \ker f'$, and $i_{f''}$ is injective, so we have 0 in M''. Then by commutativity of the diagram, $(g \circ i_f)(b) = 0$. So $i_f(b) \in \ker(g) = \operatorname{im}(\phi)$ by exactness. Then $i_f(b) = \phi(c)$ for some $c \in M'$. $i_{f'}$ is injective, so we can lift c to some $d \in \ker f'$, where $\overline{\phi}(d) = b$ by commutativity.

So the first row is exact.

Exact at $\operatorname{coker} f$:

1. $\overline{\psi} \circ \overline{h} = 0$: Let $\overline{a} \in \operatorname{coker} f'$. $\pi_{f'}$ is surjective, so $\overline{a} = \pi_{f'}(b)$ for some $b \in N'$. $\psi \circ h = 0$ by exactness, so $\psi \circ h(b) = 0$, meaning $\pi_{f''}(\psi \circ h(b)) = \overline{0}$ in $\operatorname{coker} f''$. Then $\overline{\psi} \circ \overline{h}(b) = 0$ by commutativity of the diagram, so $\overline{\psi} \circ \overline{h} = 0$.

2. $\ker\overline{\psi} \subset \mathrm{im}\overline{h}$: Let $a + \mathrm{im}f \in \ker\overline{\psi} \subset \mathrm{coker}f$. Then $a + \mathrm{im}f = \pi_f(b)$ for some $b \in N$. By commutativity of the diagram, $\psi(b) = 0$, so by exactness, $b \in \mathrm{im}h$, so $h(c) = b$ for some $c \in N'$. Then again by commutativity, $a + \mathrm{im}f \in \mathrm{im}\overline{h}$.

So the second row is exact.

Now we need to define δ and show exactness at $\ker f''$ and $\mathrm{coker}f'$. Definition of δ: Let $z \in \ker f'' \subset M''$. g is surjective, so let $g(y) = z$. Then $f(y) \in N$, and $\psi(f(y)) = 0$ by commutativity and definition of z. So by exactness, $f(y) = h(x)$ for some $x \in N'$. Define $\delta(z) = \overline{x} \in \mathrm{coker}f'$.

Well-definedness of δ: Suppose $y' \in M$ with $g(y) = i_{f''}(z)$. Then $y - y' \in \ker g = \mathrm{im}\phi$, so $y - y' = \phi(a)$ for some $a \in M'$. So $f(y - y') = h \circ f'(a)$ by commutativity. By above, $f(y) = h(x)$ and $f(y') = h(x')$ for some $x, x' \in N'$. So $(h \circ f')(a) = h(x - x')$, meaning $x - x' - f'(a) \in \ker h$. h is injective by exactness, so $x - x' - f'(a) = 0$, so $x - x' = f'(a)$. Then under $\pi_{f'}$, $\overline{x - x'} = \overline{0}$, so $\overline{x} = \overline{x'}$. Then δ is a well-defined map.

Show δ is an R-linear map (i.e. it is a homomorphism): Let us take $ra + sb \in \ker f'' \subset M''$, where $r, s \in R$ and $a, b \in M''$. Then tracing through the definition of δ, since g is surjective and R-linear, $r \cdot g(c) + s \cdot g(d) = ra + sb$, or $g(rc + sd) = ra + sb$. Then $f(rc + sd) \in N$, and $psi(f(rc + sd)) = 0$ by commutativity and definition of the kernel, so by exactness, $f(rc + sd) = r \cdot f(c) + s \cdot f(d)$ (R-linearity of f) $= r \cdot h(j) + s \cdot h(k) = h(rj + sk)$ for some $j, k \in N'$. Then $\delta(ra + sb) = \overline{rj + sk}$. But by tracing the path of ra through, we get $\delta ra = \overline{rj} = r \cdot \overline{j}$, and $\delta sb = \overline{sk} = s \cdot \overline{k}$. Then we conclude that δ is R-linear.

Exactness involving δ:

1. Show $\delta \circ \overline{g} = 0$: Let $y \in \ker f$. Then $\delta(\overline{g}(y)) = 0$ by the definition of δ.

2. Show $\ker\delta \subset \mathrm{im}\overline{g}$: Let $z \in \ker\delta$. $\delta(z) = \overline{0}$ in $\mathrm{coker}f'$, so δ maps z to $x \in \mathrm{im}f'$. Then $x = f'(a)$ for some $a \in M'$. So $(h \circ f')(a) = (f \circ \phi)(a) \in N$, or $h(x) = (f \circ \phi)(a)$. But by definition of δ, $h(x) = f(y)$ for some $y \in M$. Then $f(y - \phi(a)) = 0$, so $y - \phi(a) \in \ker f$. Then $z = g(y - \phi(a) + \phi(a)) = g(y - \phi(a)) + (g \circ \phi)(a)$, where the first summand is in $\mathrm{im}\overline{g}$, and the second summand is equal to 0. So $z \in \mathrm{im}\overline{g}$.

3. Show $\mathrm{im}\delta \subset \ker\overline{h}$: Let $\delta(z) \in \mathrm{im}\delta$. Then $\delta(z) = z + \mathrm{im}f'$, so $(\overline{h} \circ \delta)(z) = h(z) + \mathrm{im}f = f(y) + \mathrm{im}f$ (by definition of δ), which is in $\mathrm{im}f$, so it is equal to $\overline{0}$ in $\mathrm{coker}f$.

4. Show $\ker\overline{h} \subset \mathrm{im}\delta$: Let $\overline{x} \in \ker\overline{h}$. Then $h(x) \in \mathrm{im}f$, so $h(x) = f(y)$ for some $y \in M$. Let $g(y) = z \in M''$. Then $f''(z) = (\psi \circ h)(x) = 0$, so $z \in \ker f''$. Then $\delta(z) = z + \mathrm{im}f'$, and $\overline{x} \in \mathrm{im}\delta$.

Thus we have proved the snake lemma. We will now augment the statement of the lemma:

If ϕ is injective, then $\overline{\phi}$ is injective, and if ψ is surjective, then $\overline{\psi}$ is surjective.

Proof of the first statement: ϕ is injective, so ker$\phi = 0$. Let $a \in$ ker$\overline{\phi}$. By commutativity of the diagram, $i_f \circ \overline{\phi} = \phi \circ i_{f'}$, so $\phi \circ i_f(a) = 0$. i_f is injective, so $i_f(a) = 0$ if and only if $a = 0$. Then $\phi(a) = 0$, so $a \in$ kerϕ, which means $a = 0$ because ϕ is injective.

Proof of the second statement: We let $\overline{b} \in$ cokerf'', and we will show that $\overline{\psi}(\overline{c}) = \overline{b}$ for some $\overline{c} \in$ coker$f$. $\pi_{f''}$ is surjective, so $\pi_{f''}(b) = \overline{b}$. ψ is surjective, so $b = \psi(c)$ for some $c \in N$. Then $\pi_f(c) = \overline{c}$, and $\overline{\psi}(\overline{c}) = \overline{b}$ by commutativity of the diagram.